On perfectly $\alpha\delta$-continuous functions in topological spaces

P. Basker

Abstract

The concept of $\alpha\delta$-closed sets was introduced in the research paper “On Strongly-$\alpha\delta$-Super-Irresolute Functions In Topological Spaces. The aim of this paper is to consider and characterize $\alpha\delta$-irresolute and $\alpha\delta$-continuous functions via the concept of $\alpha\delta$-closed sets and to relate these concepts to the classes of $\alpha\delta O$-compact spaces and $\alpha\delta$-connected spaces.

Keywords: $\alpha\delta$-irresolute functions, $\alpha\delta$-continuous functions, $\alpha\delta O$-compact spaces, $\alpha\delta$-connected spaces.

1. INTRODUCTION

Closedness is a basic concept for the study and investigation of topological spaces. This concept has been generalized and studied by many authors from different points of views. In particular, introduced semi-open sets, pre-open sets, $\alpha$-open sets, $\delta$-closed sets and $\delta$-semi-closed sets, gap-closed sets respectively. Introduced $\alpha$-generalized closed (briefly $\alpha g$-closed) sets. Introduced and studied the concept of $\delta g$-closed sets, $\delta g s$-closed sets and $\delta s g$-closed sets. More recently, introduced and studied the notion of $\alpha\delta$-closed sets which is implied by that of $\delta$-closed sets and implies that of $\delta s\delta$-closed sets.

The notions of $\alpha\delta$-open sets, $\delta\alpha$-spaces, $\alpha\delta$-continuity and $\alpha\delta$-irresoluteness are also available. In this paper, we will continue the study of $\alpha\delta$-closed sets and associated function with introducing and characterizing $\alpha\delta$-continuous and $\alpha\delta$-irresolute functions. Further, we introduce the concepts of strong $\alpha\delta$-continuity, perfect $\alpha\delta$-continuity, $\alpha\delta O$-compactness and $\alpha\delta$-connectedness, and study their behaviour under $\alpha\delta$-continuous functions.

Throughout this paper, spaces $X$ and $Y$ always mean topological spaces. Let $X$ be a topological space and $A$ be a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $cl(A)$ and $int(A)$, respectively. A subset $A$ is said to be regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The $\delta$-interior [4] of a subset $A$ of $X$ is the union of all regular open sets of $X$ contained in $A$ and is denoted by $int_{\delta}(A)$. The $\delta$-interior of $A$ is called the $\delta$-open set and is denoted by $\delta\alpha$. The $\delta$-interior of $A$ is called the $\delta$-closed.

Alternatively, a set $A \subseteq (X, \tau)$ is called $\delta$-closed [4] if $A = cl_\delta(A)$ where $cl_\delta(A) = \{x \in U \in \tau \Rightarrow int(cl(U)) \cap A \neq \emptyset \}$. The family of all $\delta$-open (resp. $\delta$-closed) sets in $X$ is denoted by $\delta O(X)$ (resp. $\delta C(X)$). A subset $A$ of $X$ is called semi-open [1] (resp. $\alpha$-open [8], $\delta$-semi-open [5]) if $A \subseteq cl(int(A))$ (resp. $\delta cl(int(A))$). The family of all semi-open (resp. $\alpha$-open) sets in $X$ is denoted by $\delta O(X)$ (resp. $\alpha O(X)$) and is called the $\delta$-semi-open (resp. $\alpha$-open) semi-closedness of $\alpha\delta$-functions. The intersection of all semi-open (resp. $\alpha$-open, $\delta$-semi-closed) sets containing $A$ is called the $\delta$-semi-closure (resp. $\alpha$-closure, $\delta$-semi-closure) of $A$ and is denoted by $sc\delta(A)$ (resp. $acl(A)$, $\delta scl(A)$). Dually, $\alpha$-open (resp. $\alpha$-interior, $\alpha$-semi-interior) of $A$ is defined to be the union of all semi-open (resp. $\alpha$-open, $\delta$-semi-open) sets contained in $A$ and is denoted by $\alpha cl(A)$ (resp. $int(A)$, $\delta int(A)$). Note that $\delta scl(A) = A \cup cl(int_{\delta}(A))$ and $\delta int(A) = A \cup int_{\delta}(A)$. $A$ is $\alpha\delta$-closed if and only if $A$ is $\alpha\delta$-open.

We recall the following definition used in the sequel.

Definition 1.1. A subset $A$ of a space $X$ is said to be

(a) An $\alpha$-generalized closed [9] $(ag\delta)$-closed set if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$.

(b) An $\alpha\delta$-closed [12] set if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$.

(c) $\delta$-semi-generalized closed [13] $(\delta s g)$-closed set if $\delta scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\delta$-open in $(X, \tau)$.

(d) A generalized $\delta$-semi-closed [13] $(g \delta s)$-closed set if $\delta scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\delta$-open in $(X, \tau)$.

(e) A $\delta$-generalized closed [10] $(\delta g)$-closed set if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\delta$-open in $(X, \tau)$. 
Remark 1.2. [3] The following diagram holds for a subset $A$ in a topological space $(X, \tau)$:

$$
\begin{align*}
\delta g\text{-closed} \rightarrow & \delta s g\text{-closed} \\
\uparrow & \uparrow \\
\delta\text{-closed} \rightarrow & \delta\text{-semiclosed}
\end{align*}
$$

Definition 1.3. Cantor-Bendixson derivative [14]

Let $A$ be a subset of a topological space $X$. Its Cantor-Bendixson derivative $A'$ is defined as the set of accumulation points of $A$. In other words $A' = \{x \in X | x \in \overline{A} \setminus \{x\}\}$. Through transfinite induction, the Cantor-Bendixson derivative can be defined to any order $\alpha$, where $\alpha$ is an ordinal number. Let $A^{(0)} = A$. If $x$ is a successor ordinal, then $A^{(\alpha)} = (A^{(\alpha - 1)})'$. If $\alpha$ is a limit ordinal, then $A^{(\alpha)} = \bigcap_{\beta < \alpha} A^{(\beta)}$. The Cantor-Bendixson rank of the set $A$ is the least ordinal $\alpha$ such that $A^{(\alpha)} = A^{(\alpha + 1)}$. Note that $A' = A$ implies that $A$ is a perfect set.

2. $\alpha$-$\delta$-Interior and $\alpha$-$\delta$-Closure

Definition 2.1. Let $X$ be a topological space and let $x \in X$. A subset $N$ of $X$ is said to be an $\alpha$-$\delta$-nbhd of $x$ if there exists an $\alpha$-$\delta$-open set $G$ such that $x \in G \subseteq N$.

Definition 2.2. Let $A$ be a subset of $X$. A point $x \in A$ is said to be an $\alpha$-$\delta$-interior point of $A$ if $x$ is an $\alpha$-$\delta$-nbhd of $x$. The set of all $\alpha$-$\delta$-interior points of $A$ is called the $\alpha$-$\delta$-interior of $A$ and is denoted by $\alpha \delta int(A)$.

Definition 2.3. For a subset $A$ of $(X, \tau)$, we define the $\alpha$-$\delta$-closure of $A$ as follows:

$$
\alpha \delta cl(A) = \cap \{F | F \text{ is } \alpha \delta \text{-closed in } A, F \subseteq C\}.
$$

Lemma 2.4. Let $A$ be a subset of $(X, \tau)$ and $x \in X$. Then $x \in \alpha \delta cl(A)$ if and only if $V \cap A \neq \emptyset$ for every $\alpha \delta$-open set $V$ containing $x$.

Proof. Suppose that there exists a $\alpha \delta$-open set $V$ containing $x$ such that $V \cap A = \emptyset$. Since $A \subseteq X$, $\alpha \delta cl(A) \subseteq V \cap X$ and then $x \notin \alpha \delta cl(A)$, a contradiction. Conversely, suppose that $x \notin \alpha \delta cl(A)$. Then there exist an $\alpha \delta$-closed set $F$ containing $A$ such that $x \notin F$. Since $x \in X \setminus F$ and $X \setminus F$ is $\alpha \delta$-open,$(X \setminus F) \cap A = \emptyset$, a contradiction.

Lemma 2.5. Let $A$ and $B$ be subsets of $(X, \tau)$. Then we have

(a) $\alpha \delta cl(\emptyset) = \emptyset$ and $\alpha \delta cl(X) = X$.
(b) If $A$ is $\alpha \delta$-closed, then $\alpha \delta cl(A) = A$.
(c) If $A \subseteq B$, then $\alpha \delta cl(A) \subseteq \alpha \delta cl(B)$.
(d) $\alpha \delta cl(A) = \alpha \delta cl(\alpha \delta int(A))$.

Proof. Straightforward.

Lemma 2.6. Let $(X, \tau)$ be a topological space and $A \subseteq X$, $B \subseteq X$. The following properties hold:

(a) $\alpha \delta int(\emptyset) = \emptyset$ and $\alpha \delta int(X) = X$.
(b) $\alpha \delta int(A) \subseteq A$.
(c) $\alpha \delta int(A) = \alpha \delta int(\alpha \delta int(A))$.
(d) If $A \subseteq B$, then $\alpha \delta int(A) = \alpha \delta int(B)$.
(e) If $B$ is any $\alpha \delta$-open set contained in $A$, then $B \subseteq \alpha \delta int(A)$.

Proof. Straightforward.

Remark 2.7. (a) If $A$ is $\alpha \delta$-closed in $(X, \tau)$, then $\alpha \delta cl(A) = A$. But the converse is not true which shows from the following example: Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, d\}\}$.

Let $A = \{b, d\}$. Then $\alpha \delta cl(A) = Abut A is not $\alpha \delta$-closed.
(b) In general, $\alpha \delta cl(A \cap B) \subseteq \alpha \delta cl(A) \cap \alpha \delta cl(B)$. For example, let $X = \{-1, 1\}$ and $\tau = \{\emptyset, \{-1, 1\}, \{0\}\}$. Let $A = \{-1, 0\}$ and $B = \{0, 1\}$. Since both $A$ and $B$ are not $\alpha \delta$-closed, $\alpha \delta cl(A) = \{-1\}$ and $\alpha \delta cl(B) = \{1\}$. Then $\alpha \delta cl(A \cap B) = \{0\}$ but $\alpha \delta cl(A) \cap \alpha \delta cl(B) = \emptyset$.

Theorem 2.8. Let $A$ be any subset of $X$.

(a) $(\alpha \delta int(A))^c = \alpha \delta cl(A)^c$.
(b) $\alpha \delta int(A) = (\alpha \delta cl(A)^c)^c$.
(c) $\alpha \delta cl(A) = (\alpha \delta int(A)^c)^c$.

Proof. Let $x \in (\alpha \delta int(A))^c$. Then $x \notin \alpha \delta int(A)$, i.e., every $\alpha \delta$-open set $U$ containing $x$ is such that $U \nsubseteq A$. Then every $\alpha \delta$-open set $U$ containing $x$ is such that $U \cap A^c = \emptyset$. By Lemma 2.4, $x \in \alpha \delta cl(A)^c$ and therefore $(\alpha \delta int(A))^c \subseteq \alpha \delta cl(A)^c$. Conversely, let $x \in \alpha \delta cl(A)^c$. Then by Lemma 2.4, every $\alpha \delta$-open set $U$ containing $x$ is such that $U \cap A^c = \emptyset$.

That is every $\alpha \delta$-open set $U$ containing $x$ is such that $U \nsubseteq A$. This implies by definition of $\alpha \delta$-interior of $A$, $x \notin \alpha \delta int(A)$. That is $x \in (\alpha \delta int(A))^c$ and $\alpha \delta cl(A)^c \subseteq (\alpha \delta int(A))^c$. Thus $(\alpha \delta int(A))^c = (\alpha \delta cl(A))^c$.

(b) Follows by taking complements in (a).

(c) Follows by replacing $A$ by $A^c$ in (a).

Definition 2.9. For a subset $A$ of $(X, \tau)$

(a) $Int_\delta(A) = \{F : Fis \delta - \text{open in } X, F \subseteq A\}$.
(b) $\delta gs\text{-int}(A) = \{F : Fis \delta gs - \text{open in } X, F \subseteq A\}$.
(c) $\delta g\text{-int}(A) = \{F : Fis \delta \text{-open in } X, F \subseteq A\}$.

Theorem 2.10. If $A$ is a subset of $X$, Then $\alpha \delta int(A) = \{F : F is \alpha \delta - \text{open in } X, F \subseteq A\}$.

Proof. If $A$ be a subset of $X$, Then $x \in \alpha \delta int(A) \Rightarrow x$ is a $\alpha \delta$-interior point of $A$.

If $x$ is a $\alpha \delta$-nbhd of point $x$.

$\Rightarrow$ there exists $\alpha \delta$-open set $F$ such that $x \in F \subseteq A$.

$\Rightarrow x \in \{F : F is \alpha \delta - \text{open in } X, F \subseteq A\}$.

$\Rightarrow x \in \alpha \delta int(A) = \{F : F is \alpha \delta - \text{open in } X, F \subseteq A\}$.

Theorem 2.11. If $A$ is a subset of $X$, Then

(a) $Int_\delta(A) \subseteq \alpha \delta int(A)$.
(b) $\alpha \delta int(A) \subseteq \delta gs\text{-int}(A)$.
(c) $\alpha \delta int(A) \subseteq g\delta s\text{-int}(A)$.

Proof. (a) Let $A$ be a subset of $X$. Then $x \in Int_\delta(A) \Rightarrow x \in \{F \subseteq X : F is \delta - \text{open, } F \subseteq A\}$.

$\Rightarrow$ There exists $\delta$-open set $F$ such that $x \in F \subseteq A$.

$\Rightarrow$ There exists $\alpha \delta$-open set $F$ such that $x \in F \subseteq A$, as every $\delta$-open set is a $\alpha \delta$-open set in $X$.

$\Rightarrow x \in \{F \subseteq X : F is \alpha \delta - \text{open, } F \subseteq A\}$.

Hence $Int_\delta(A) \subseteq \alpha \delta int(A)$.

The proof of (b) and (c) follows from (a). Since we know that every $\alpha \delta$-open set is a $\delta gs\text{-open}$ set and every $\alpha \delta$-open set is a $g\delta s\text{-open}$ set. Containment relation in the above may be proper as seen from the following example.
Example 2.12. (a) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b, a\}, \{a, b, c\}\}$. Let $a \in \text{Int}_S(a) = \emptyset$ and $\text{Int}_S(a) = \emptyset$. It follows that $\text{Int}_S(a) \subseteq a \text{Int}_S(a)$ and $\text{Int}_S(a) \neq a \text{Int}_S(a)$.

(b) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c, a\}, \{c, a, d\}\}$. Let $a \in \text{Int}_S(b)$ and $\text{Int}_S(b) \neq a \text{Int}_S(b)$.

(c) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b, a\}, \{a, c\}\}$. Let $a \in \text{Int}_S(b)$ and $\text{Int}_S(b) \neq a \text{Int}_S(b)$.

Definition 2.13. The union of all $\pi g$-open sets, each contained in a set $S$ in a topological space $X$ is called the $\pi g$-interior of $S$ and is denoted by $\pi g$-Int$(S)$.

Remark 2.14. For a subset $A$ of a space, $a \text{Int}_S(A) \subset \pi g$-Int$(A)$.

Definition 2.15. Let $(X, \tau)$ be a topological space and $S \subset X$. The set $\tau \cap \{A: S \subset A \in \pi g$-closed $\}$ is called the $\delta$-semi-generalized closure of $S$ and is denoted by $\pi g$-cl$(S)$.

Definition 2.16. For a topological space $(X, \tau)$,

(a) $\pi g$-cl$(X \setminus \tau) = \{U \subset X \mid a \text{cl}_C(X \setminus U) = X \setminus U\}$. 

(b) $\pi g$-cl$(X \setminus \tau) = \{U \subset X \mid a \text{cl}_C(X \setminus U) = X \setminus U\}$. 

(c) $\pi g$-cl$(X \setminus \tau) = \{U \subset X \mid a \text{cl}_C(X \setminus U) = X \setminus U\}$. 

Definition 2.17. A space $(X, \tau)$ is called

(a) $T^\delta$-space if every $\delta$-closed set in it is closed.

(b) Submaximal if every compact subset is open.

(c) $\pi g$-cl$(X \setminus \tau)$-space if every $\pi g$-closed set in it is isolated.

Proposition 2.18. If $\delta X \setminus \tau)$ is closed under finite intersections, then $\pi g$-cl$(X \setminus \tau)$ is a topology for $X$.

Proof. Clearly, $\emptyset, X \in \pi g$-cl$(X \setminus \tau)$. If $X \setminus \tau = \{\alpha \in X \mid a \text{cl}_C(\{\alpha\}) = \{\alpha\}\}$, then $a \text{cl}_C(\{\alpha\}) = a \text{cl}_C(\{\alpha\})$ and hence $A \subset X \setminus \tau$. Let $\alpha \text{cl}_C(\{\alpha\}) = X \setminus \tau$. Therefore, $\pi g$-cl$(X \setminus \tau)$ is a topology.

Proposition 2.19. For a subset $A$ of $(X, \tau)$, the following statements hold:

(a) $A \in \pi g$-cl$(A) \subset a \text{cl}_C(A) \subset \delta g$-cl$(A)$. 

(b) $\pi g$-cl$(A) \subset a \text{cl}_C(A) \subset \delta g$-cl$(A)$. 

Proof. The proof follows from the definitions.

Definition 2.20. A subset $A$ of a space $X$ is called generalized $\delta p$-closed (briefly, $g\delta p$-closed) if $\delta -p\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is regular open in $(X, \tau)$.

Definition 2.21. For a subset $A$ of a topological space $(X, \tau)$, $g\delta p r -cl(A) = \pi \tau : F \in \pi \tau, F$ is $g\delta p r -closed \ in \ X)$. 

Definition 2.22. The intersection of all $\pi g$-closed sets, each containing a set $S$ in a topological space $X$ is called the $\pi g$-closure of $S$ and is denoted by $\pi g$-cl$(S)$.

Remark 2.23. For a subset $A$ of a space, $g\delta p r$-cl$(A) \subset a \text{cl}_C(A)$ and $\pi g$-cl$(A) \subset a \text{cl}_C(A)$.

Theorem 2.24. Let $(X, \tau)$ be a space. Then

(a) Every $\alpha$-closed set is $\delta$-closed (i.e., $(X, \tau)$ is $T_\alpha$-space) if and only if $\pi g$-cl$(X, \tau)$.

(b) Every $\alpha$-closed set is closed if and only if $\pi g$-cl$(X, \tau)$.

(c) Every $\delta$-closed set is $\alpha$-closed (i.e., $\delta g$-cl$(X, \tau)$).

Proof. (a) Let $A \in T_\alpha$. Then $\alpha \text{cl}_C(X \setminus A) = X \setminus A$. By hypothesis, $\delta g$-cl$(X \setminus A) = X \setminus A$ and hence $A \subset X \setminus A$. Conversely, let $A$ be $\alpha$-closed. Then $\alpha \text{cl}_C(X \setminus A) = X \setminus A$ and hence $X \setminus A \in T_\alpha$. Therefore, $\alpha \text{cl}_C(X \setminus A)$ is $\alpha$-closed.

(b) Similar to (a).

(c) Let $A \in T_\alpha$. Then $\delta g$-cl$(X \setminus A) = X \setminus A$ and by hypothesis, $\alpha \text{cl}_C(X \setminus A) = \delta g$-cl$(X \setminus A) = X \setminus A$.

Hence $\pi g$-cl$(X, \tau)$, i.e., $\pi g$-cl$(X)$.

Theorem 2.25. Given a space $X$, it follows that

(a) If $\pi g$-cl$(X, \tau)$ is $\delta O(X)$, then $\pi g$-closed $\pi g$-cl$(X, \tau)$.

(b) If $X$ is maximal and $\pi g$-cl$(X, \tau)$, then $\pi g$-cl$(X, \tau)$.

![](Image)

3. Strongly $\alpha$-continuous and Perfectly $\alpha$-continuous functions.

Definition 3.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

(a) $\alpha$-continuous if $f^{-1}(V)$ is an $\alpha$-closed set in $(X, \tau)$ for every closed set $V \subset Y$.

(b) $\alpha$-irresolute if $f^{-1}(V)$ is an $\alpha$-closed set in $(X, \tau)$ for every closed set $V \subset Y$.

(c) Strongly $\alpha$-continuous (briefly, $S_\alpha$-super-irr) if $f^{-1}(V)$ is an $\alpha$-closed set in $(X, \tau)$ for every $\alpha$-closed set $V \subset Y$.

(d) Super $\alpha$-super-irresolute if $f^{-1}(V)$ is an $\alpha$-closed set in $(X, \tau)$ for every $\alpha$-closed set $V \subset Y$.

(e) Super $\alpha$-continuous (Nairi T., [1979]) if $f^{-1}(V)$ is $\delta$-closed in $(X, \tau)$ for every $\delta$-closed set $V \subset Y$.

Remark 3.2. (a) If $\pi g$-cl$(X, \tau)$, then $\pi g$-continuity and $\alpha$-continuity coincide.

(b) Every $\alpha$-continuous function defined on $\pi g$-cl$(X, \tau)$ is $\pi g$-open.

Definition 3.3. A subset $A$ of a space $(X, \tau)$ is said to be $g\delta p$-closed if $\delta p$-cl$(A) \subset U$ whenever $A \subset U$ and $U$ is $\pi g$-open in $X$.

Definition 3.4. A subset $A$ of a space $(X, \tau)$ is said to be $\pi gp$-closed if $p\text{cl}(A) \subset U$ whenever $A \subset U$ and $U$ is $\pi g$-open in $X$.

Definition 3.5. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $g\delta p$-continuous (resp. $g\delta pr$-continuous) if $f^{-1}(V)$ is $g\delta p$-closed (resp. $g\delta pr$-closed) in $X$ for every closed set $V \subset Y$.

Definition 3.6. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be almost $g\delta p$-continuous if $f^{-1}(int\{cl(V)\}) \subset gp$-open for every $V \subset \sigma$.

Remark 3.7. Every $\alpha$-continuous function is $g\delta p$-continuous and so $g\delta pr$-continuous and almost $gp$-continuous.

Theorem 3.8. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function.

(a) The following statements are equivalent:
(i) $f$ is $\alpha\delta$-continuous.
(ii) The inverse image of every open set in $Y$ is $\alpha\delta$-open in $X$.
(iii) If $f : (X, \tau) \to (Y, \sigma)$ is $\alpha\delta$-continuous, then $f(\alpha\delta(\sigma(A))) \subseteq \alpha\delta(f(A))$ for every subset $A$ of $X$.

The following statements are equivalent:

(i) For each $x \in X$ and each open set $V$ containing $f(x)$, there exists a $\alpha\delta$-open set $U$ containing $x$ such that $f(U) \subseteq V$.
(ii) For every subset $A$ of $X$, $f(\alpha\delta(A)) \subseteq \alpha\delta(f(A))$.
(iii) Suppose $\tau_{\alpha\delta}$ is a topology. The function $f : (X, \tau_{\alpha\delta}) \to (Y, \sigma)$ is $\alpha\delta$-continuous.

**Proof:**

(a) Straightforward.

(b) Let $A \subseteq X$. Since $f$ is $\alpha\delta$-continuous and $A \subseteq f^{-1}(\alpha\delta(f(A)))$, we obtain $\alpha\delta(A) \subseteq f^{-1}(\alpha\delta(f(A)))$ and then $f(\alpha\delta(A)) \subseteq f(\alpha\delta(f(A)))$.

(c) $(i) \Rightarrow (ii)$ Let $y \notin f(\alpha\delta(A))$ and let $V$ be any open neighborhood of $y$. Then there exists an $x \in X$ and a $\alpha\delta$-open set $U$ such that $f(x) = y$, $x \in U$, $x \in \alpha\delta(A)$, and $f(U) \subseteq V$. By Lemma 2.4, $\tau_X \cap \Phi = \emptyset$, and hence $f(A) \cap V = \emptyset$. Therefore, $y = f(x) \notin f(A)$.

(ii) $(i) \Rightarrow (iii)$ Let $x \in X$ and $V$ be any open set containing $f(x)$. Let $A = f^{-1}(V)$. Since $f(\alpha\delta(A)) \subseteq f(f(A))$, then $\alpha\delta(A) = A$. Since $x \in \alpha\delta(A)$, there exists a $\alpha\delta$-open set $U$ containing $x$ such that $U \cap A = \emptyset$ and hence $f(U) \subseteq f(X \setminus A) \subseteq V$.

**Example 3.12.** (a) Let $X = \{a, b, c\}$ and $\mathcal{A} = \{X, \{a, b, c\}, \{b, c\}, \{a\}, \emptyset\}$ and $\mathcal{B} = \{X, \{a, b\}, \{b, c\}, \{a, c\}\}$ and $\mathcal{C} = \{X, \{a, b\}, \{a, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity. Then $f^{-1}(\{b, c\})$ is $\alpha\delta$-irresolute but not a $\mathcal{B}\delta$-super-irr, since $f^{-1}(\{b, c\})$ is not a $\delta$-closed set in $(X, \tau)$.

(b) Let $X = \{a, b, c\}$ and $\mathcal{A} = \{X, \{a, b\}, \{b, c\}, \{a, c\}\}$ and $\mathcal{B} = \{X, \{a, b\}, \{a, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity. Then $f^{-1}(\{b, c\})$ is $\alpha\delta$-continuous but not $\alpha\delta\delta$-irresolute. Since $f^{-1}(\{c\})$ is not $\alpha\delta$-closed in $(X, \tau)$.

**Definition 3.13.** A function $f : (X, \tau) \to (Y, \sigma)$ is called Strongly-\(\alpha\delta\)-continuous (briefly ST-\(\alpha\delta\)-Continuous) if $f^{-1}(V)$ is an open set in $(X, \tau)$ for every $\alpha\delta$-open set $V$ of $(Y, \sigma)$.

**Definition 3.14.** A function $f : (X, \tau) \to (Y, \sigma)$ is called Perfect-\(\alpha\delta\)-continuous (briefly P\(\alpha\delta\)-Continuous) if $f^{-1}(V)$ is a clopen set in $(X, \tau)$ for every $\alpha\delta$-open set $V$ of $(Y, \sigma)$.

Note that a function $f : (X, \tau) \to (Y, \sigma)$ is Perfect-\(\alpha\delta\)-continuous if and only if the inverse image of every $\alpha\delta$-closed of $Y$ is clopen in $X$.

**Theorem 3.15.** If the function $f : (X, \tau) \to (Y, \sigma)$ is $ST-\alpha\delta$-Continuous and the function $g : (Y, \sigma) \to (Z, \eta)$ are $\alpha\delta$-continuous, then $g \circ f : (X, \tau) \to (Z, \eta)$ is continuous.

**Proof.** Follows from Definitions 3.1 (a) and Definition 3.13.

**Theorem 3.16.** If $f : (X, \tau) \to (Y, \sigma)$ is perfectly continuous and if $Y$ is both $\delta\delta$-space and $T\delta\delta$-space, then $f$ is $PER-\alpha\delta$-Continuous.

**Proof.** Follows from Theorem 2.25.

4. $\alpha\delta\delta$-compact spaces and $\alpha\delta$-connected spaces.

**Definition 4.1.** A topological space $(X, \tau)$ is called $\alpha\delta\delta$-compact if for every cover of $X$ by $\alpha\delta\delta$-open sets ($\alpha\delta\delta$-open cover) has finite subcover.

**Definition 4.2.** [16] A topological space $X$ is called $\pi\gamma$-compact if for every cover of $X$ by $\pi\gamma$-open sets has finite subcover.

**Remark 4.3.** Every $\pi\gamma$-compact space is $\alpha\delta\delta$-compact.

**Definition 4.4.** A subset $A$ of a space $(X, \tau)$ is called $\alpha\delta\delta$-compact relative to $X$ if for every collection $\{U_i : i \in A\}$ of $\alpha\delta\delta$-open subsets of $X$ such that $A \subseteq \bigcup \{U_i : i \in A\}$, there exists a finite subset $A_0$ of $A$ such that $A \subseteq \bigcup \{U_i : i \in A_0\}$.

**Definition 4.5.** A subset $A$ of a space $(X, \tau)$ is called $\alpha\delta$-compact if $A$ is $\alpha\delta\delta$-compact as a subspace of $X$.

**Lemma 4.6.** A topological space $(X, \tau)$ is strongly compact if and only if $X$ is compact and the Cantor-Bendixon derivative of $X/\tau$ is finite.

**Example 4.7.** Let $Z$ be the set of all integers and let $\tau$ be the double point excluded topology on $Z$, i.e., $\tau = \{U \subseteq Z : U \cap \{0, 1\} = \emptyset\}$. Since this is a compact topological space with finite Cantor-Bendixon derivative, by Lemma 4.6., $(Z, \tau)$ is strongly...
compact. To see that \((Z, \tau)\) is not \(a\delta\delta\)-compact, we consider the following \(a\delta\)-open cover of \(Z\): \(A = \{(1, x) : x \in Z \text{ and } x \neq 0\}\) \(\cup\) \(\{0\}\). It is obvious that \(A\) has no finite subcover.

**Theorem 4.8.**
(a) Every \(a\delta\)-closed subsets of a \(a\delta\delta\)-compact space \(X\) is \(a\delta\delta\)-compact relative to \(X\).
(b) The surjective \(a\delta\)-continuous image of a \(a\delta\delta\)-compact space is compact.
(c) If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is \(a\delta\)-irresolute and a subset \(A\) of \(X\) is \(a\delta\delta\)-compact relative to \(X\), then its image \(f(A)\) is \(a\delta\delta\)-compact relative to \(Y\).
(d) If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is strongly \(a\delta\)-continuous surjective and \(X\) is compact, then \(Y\) is \(a\delta\delta\)-compact.

**Proof:** (a) Let \(A\) be a \(a\delta\)-closed subset of a \(a\delta\delta\)-compact space \((X, \tau)\). Let \(\{U_i : i \in I\}\) be a cover of \(A\) by \(a\delta\)-open subsets of \(X\). So \(A \subset \bigcup_{i \in I} U_i\) and then \((X/A) \cup (U_{i \in I \in A}) = X\). Since \(X\) is \(a\delta\delta\)-compact, there exists a finite subset \(I_0\) of \(I\) such that \((X/A) \cup (U_{i \in I_0}) = X\). Then \(A \subset \bigcup_{i \in I_0} U_i\) and hence \(A\) is \(a\delta\delta\)-compact relative to \(X\).

(b) Let \((X, \tau)\) be a \(a\delta\delta\)-compact space and \((Y, \sigma)\) be a \(a\delta\delta\)-continuous function. Let \(\{U_i : i \in I\}\) be a cover of \(X\) by open sets. Then \(f^{-1}\{(U_i) : i \in I\}\) is a cover of \(X\) by \(a\delta\delta\)-open sets, since \(f\) is \(a\delta\delta\)-continuous. By \(a\delta\delta\)-compactness of \(X\), there is a finite subset \(I_0\) of \(I\) such that \(X = \bigcup_{i \in I_0} f^{-1}\{(U_i)\} \). Since \(f\) is surjective, \(Y = \bigcup_{i \in I_0} U_i\) and hence \(Y\) is compact.

(c) and (d) are similar to (b).

**Lemma 4.9.** Let \(p : X \times Y \rightarrow X\) be a projection. If \(A\) is \(a\delta\)-closed subset of \(X\), then \(p^{-1}(A) = A \times Y\) is \(a\delta\)-closed in \(X \times Y\).

**Proof:** Let \(A \times Y \subset U\) and \(U\) be an \(a\delta\)-open subset of \(X \times Y\). Then \(U = U \times Y\) for some \(a\delta\)-open set of \(X\). Since \(A\) is \(a\delta\)-closed in \(X\), \(\delta \cap \delta(\bar{A}) \subset U\), and so \(\delta \cap \delta(\bar{A}) \times Y \subset V \times Y = U\), i.e., \(\delta \cap \delta(\bar{A} \times Y) \subset U\). Hence \(A \times Y\) is \(a\delta\)-closed in \(X \times Y\).

**Theorem 4.10.** If the product space of two non-empty topological spaces is \(a\delta\delta\)-compact, then each factor space is \(a\delta\delta\)-compact.

**Proof:** Let \(X \times Y\) be the product space of the non-empty spaces \(X\) and \(Y\). By Lemma 4.9, the projection \(p : X \times Y \rightarrow X\) is \(a\delta\delta\)-irresolute. By Proposition 4.14(b), the \(a\delta\delta\)-irresolute image \(p(X \times Y) = X\) of the \(a\delta\delta\)-connected space \(X \times Y\), is \(a\delta\delta\)-connected. The proof for the space \(Y\) is similar to the case of \(X\).

**Definition 4.11.** A space \((X, \tau)\) is said to be \(a\delta\)-connected if \(X\) cannot be expressed as the disjoint union of two nonempty \(a\delta\)-open sets. A subset of \(X\) is \(a\delta\)-connected if it is \(a\delta\)-connected as a subspace.

**Proposition 4.12.** For a space \((X, \tau)\) the following are equivalent:
(a) \(X\) is \(a\delta\)-connected.
(b) The only subsets of \(X\) which are both \(a\delta\)-open and \(a\delta\)-closed are the empty set \(\emptyset\) and \(X\).
(c) Every \(a\delta\)-continuous function of \(X\) into a discrete space \(Y\) with at least two points is a constant function.

**Proof:** The proof is similar to that of Proposition 6.2 of [Ganjamalal, Y et al. 1999].

**Proposition 4.13.**
(a) If \((X, \tau)\) is a space with \(\tau^* = \tau\), then \(X\) is connected if and only if \(X\) is \(a\delta\)-connected.
(b) If \((X, \tau)\) is a space with \(\tau^* = \tau^\#\), then \(X\) is connected if and only if \(X\) is \(g\delta\delta\)-connected.

**Proof:** (a) Follows from the definitions and Theorem 2.24(b).
(b) Follows from the definitions and Theorem 2.24(c).

**Proposition 4.14.**
(a) If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is \(a\delta\)-continuous surjective and \(X\) is \(a\delta\)-connected, then \(Y\) is connected.
(b) If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is \(a\delta\)-irresolute surjective and \(X\) is \(a\delta\)-connected, then \(Y\) is \(a\delta\)-connected.
(c) Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a \(a\delta\)-continuous surjection and \(a\delta\delta\delta(X, \tau)\) is closed under finite intersections. If \(H\) is \(ag\)-open, \(a\delta\delta\)-closed and \(a\delta\delta\)-connected, then \(f(H)\) is a connected.

**Proof:** (a) and (b) follow from definitions.
(c) By Proposition 3.10, the restriction \(f|\cap(H)\) is \(a\delta\delta\)-continuous. By (a), the image of the \(a\delta\delta\delta\)-connected space \((H, \tau/H)\) under \(f|\cap(H) : (X, \tau/H) \cap (f(H), \sigma/f(H))\) is connected. Hence \(f(H)\) is connected subset of \(Y\).

**Theorem 4.15.** If the product space of two non-empty spaces is \(a\delta\delta\)-connected, then each factor space is \(a\delta\delta\)-connected.

**Proof:** Let \(X \times Y\) be the product space of non-empty spaces \(X\) and \(Y\). By Lemma 4.9, the projection \(p : X \times Y \rightarrow X\) is \(a\delta\delta\)-irresolute. By Proposition 4.14(b), the \(a\delta\delta\)-irresolute image \(p(X \times Y) = X\) of the \(a\delta\delta\)-connected space \(X \times Y\), is \(a\delta\delta\)-connected. The proof for the space \(Y\) is similar to the case of \(X\).

**Conclusion**
In this paper we have studied about \(a\delta\)-open and \(a\delta\)-closed sets. Then we introduced \(a\delta\)-irresolute and \(a\delta\)-continuous functions via the concept of \(a\delta\)-closed sets and to relate these concepts to the classes of \(a\delta\delta\delta\)-compact spaces and \(a\delta\delta\)-connected spaces.

**REFERENCES:**


